A priori Estimates

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In previous lectures we considered the maximum principle for homogeneous equations. We will now consider supremum estimates in the case of inhomogeneous equations.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu \geq f \text{ in } \Omega$$

for some functions a^{ij}, b^i, c, and f on Ω. Suppose L is an elliptic operator,

$$\beta = \sup_\Omega \frac{|b^i|}{\lambda} < \infty,$$

and $c \leq 0$ in Ω. Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \sup_\Omega \frac{|f^-|}{\lambda}$$

for $C = e^{(\beta+1)d} - 1$, where $d = \operatorname{diam} \Omega$ and $f^- = \max\{-f, 0\}$. Note that if $Lu = f$ in Ω, we have

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C \sup_\Omega \frac{|f|}{\lambda}$$

for $C = e^{(\beta+1)d} - 1$.

Proof. Without loss of generality let Ω lie between the slab $0 < x_1 < d$. Set $L_0 = a^{ij} D_{ij} + b^i D_i$. Let

$$v = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup \frac{|f^-|}{\lambda},$$

where $\alpha \geq \beta + 1$. We claim that $Lu \geq f \geq Lv$ in Ω, in which case we can apply the comparison principle using the fact that $u \leq v$ on $\partial\Omega$ to conclude that $u \leq v$ on $\overline{\Omega}$. For $\alpha \geq \beta + 1$ we have

$$L_0 e^{\alpha x_1} = (\alpha^2 a^{11} + \alpha b^1) e^{\alpha x_1} \geq \lambda(\alpha^2 - \alpha\beta) e^{\alpha x_1} \geq \lambda.$$

Thus

$$\begin{aligned} Lv &= L_0 v + cv \leq L_0 v && \text{(since } c \leq 0,\ v \geq 0 \text{ in } \Omega) \\ &= -L_0(e^{\alpha x_1}) \sup \frac{|f^-|}{\lambda} && \text{(by linearity of } L) \\ &\leq -\lambda \sup \frac{|f^-|}{\lambda} && \text{(since } L_0(e^{\alpha x_1}) \geq \lambda \text{ in } \Omega) \\ &\leq f \end{aligned}$$

in Ω. By the comparison principle, $u \leq v$ on Ω. In particular,

$$\sup_\Omega u \leq \sup_\Omega v = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - 1) \sup \frac{|f^-|}{\lambda}.$$

Replacing u with $-u$ completes the proof in the case that $Lu = f$ in Ω. □

Corollary 1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega$$

for some functions a^{ij}, b^i, c, and f on Ω. Suppose L is an elliptic operator and

$$\beta = \sup_\Omega \frac{|b^i|}{\lambda} < \infty.$$

Suppose that Ω is a small enough domain that

$$\gamma = (e^{(\beta+1)d} - 1)\frac{c_+}{\lambda} < 1,$$

where $d = \operatorname{diam} \Omega$ and $c = c_+ - c_-$ for $c_+ = \max\{c, 0\}$ and $c_- = \max\{-c, 0\}$. Then

$$\sup_\Omega u \leq \frac{1}{1-\gamma}\left(\sup_{\partial\Omega} u^+ + C \sup_\Omega \frac{|f^-|}{\lambda}\right)$$

for some constant $C \in (0, \infty)$ depending only on β and d.

Proof. Observe that

$$a^{ij} D_{ij} u + b^i D_i u - c_- u \geq f - c_+ u \text{ in } \Omega.$$

By Theorem 1,

$$\begin{aligned} \sup_\Omega |u| &\leq \sup_{\partial\Omega} |u| + C \sup_\Omega \frac{|f|}{\lambda} + C \sup_\Omega \frac{c_+}{\lambda} \sup_\Omega |u| \\ &\leq \sup_{\partial\Omega} |u| + C \sup_\Omega \frac{|f|}{\lambda} + \gamma \sup_\Omega |u|. \end{aligned}$$

Since $\gamma < 1$,

$$\sup_\Omega |u| \leq \frac{1}{1-\gamma}\left(\sup_{\partial\Omega} u^+ + C \sup_\Omega \frac{|f^-|}{\lambda}\right).$$

□

Corollary 2. *(Uniqueness of Solutions to the Dirichlet Problem on Small Domains) Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the Dirichlet problem*

$$\begin{aligned} &Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega, \\ &u = \varphi \text{ on } \partial\Omega, \end{aligned}$$

for some functions a^{ij}, b^i, c, and f on Ω and $\varphi \in C^0(\partial\Omega)$ such that L is an elliptic operator and

$$\beta = \sup_\Omega \frac{|b^i|}{\lambda} < \infty, \quad \sup_\Omega \frac{|c|}{\lambda} < \infty.$$

Suppose that Ω *is a small enough domain that*

$$\gamma = (e^{(\beta+1)d} - 1)\frac{c_+}{\lambda} < 1,$$

where $d = \operatorname{diam}\Omega$ *and* $c = c_+ - c_-$ *for* $c_+ = \max\{c, 0\}$ *and* $c_- = \max\{-c, 0\}$*. Then there is at most one solution* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *to the Dirichlet problem (i.e. there may be no solution or a unique solution but there cannot be two or more solutions).*

Proof. Suppose u_1 and u_2 are two solutions to the Dirichlet problem. Then

$$\begin{aligned} L(u_1 - u_2) &= 0 \text{ in } \Omega, \\ u_1 - u_2 &= 0 \text{ on } \partial\Omega. \end{aligned}$$

By Corollary 1, $u_1 - u_2 = 0$ on $\overline{\Omega}$, i.e. $u_1 = u_2$ on $\overline{\Omega}$. □

References: Gilbarg and Trudinger, Section 3.3.